STRONG AND WEAK CONVERGENCE OF POPULATION SIZE IN SUPERCRITICAL CATALYTIC BRANCHING PROCESS

Ekaterina Vl. Bulinskaya[1,2]

Lomonosov Moscow State University

Abstract

A general model of catalytic branching process (CBP) with any finite number of catalysis centers in a discrete space is studied. More exactly, it is assumed that particles move in this space according to a specified Markov chain and they may produce offspring only in the presence of catalysts located at fixed points. The asymptotic (in time) behavior of the total number of particles as well as the local particles numbers is investigated. The problems of finding the global extinction probability and local extinction probability are solved. Necessary and sufficient conditions are established for phase of pure global survival and strong local survival. Under wide conditions the limit theorems for the normalized total and local particles numbers in supercritical CBP are proved in the sense of almost surely convergence as well as with respect to convergence in distribution. Generalizations of a number of previous results are obtained as well. In the proofs the main role is played by recent results by the author devoted to classification of CBP and the moment analysis of the total and local particles numbers in CBP.

Keywords and phrases: catalytic branching process, extinction probability, strong local survival, pure global phase, limit theorems, multi-type Bellman-Harris process.

2010 *AMS classification*: 60J80, 60J27.

Theory of branching processes is a classical part of probability theory (see, e.g., [12]). Its origin goes back to the middle of XIX century when F.Galton and G.W.Watson proposed a model elucidating the extinction of family names. In the framework of this model the primary branching process called Galton-Watson process arose. It was found that the value of extinction probability as well as the mean population size depend essentially on whether the mean offspring number of a population representative is greater than, equal to or less than 1. Galton-Watson branching process having that mean offspring number is called supercritical, critical and subcritical, respectively. Only for supercritical Galton-Watson process the extinction probability is less than 1 whereas the mean population size tends to infinity as time grows. Therefore, after solving the problem of finding the extinction probability the question arises how fast the population grows in the case of its survival. The answer to that question is given by limit theorems for the population size. The results related to the Galton-Watson process are contained, e.g., in [14], sections 1-6 and 9.

We are interested in a more involved model than the Galton-Watson process, namely catalytic branching process (CBP) with an arbitrary finite number of catalysis centers. Its specifics

[1] *Email address:* `bulinskaya@yandex.ru`

[2] The work is partially supported by Dmitry Zimin Foundation “Dynasty” and RFBR grant 14-01-00318.

consists in possibility for the population representatives (which further will be called particles) not only to produce offsprings but also to move in the space. Moreover, we assume that the particles produce offsprings exclusively in the presence of catalysts which are located in a finite set of the space points. For CBP it is natural to raise the question not only on the global particles extinction but also on the local extinction as well as to investigate the asymptotic behavior of total and local particles numbers. The present paper is devoted to these problems. We consider a discrete space where the particles move, although branching processes in a continuous space are of doubtless interest as well (see, e.g., [3]).

The study of general model of CBP with any finite number of catalysis centers in a discrete space is initiated in [4]. Particular cases of this model were analyzed in many publications, e.g., in [1], [7], [9], [13] and others. In dependence on the value of the Perron root of a certain matrix a classification of CBP as supercritical, critical and subcritical processes was proposed in [4]. The naturalness of the classification was confirmed by implemented by the author asymptotic analysis of the moments of the total and local particles numbers existing in the process at time t, as $t \to \infty$. In the present work the problem of finding the probabilities of global and local population extinction is solved and new limit theorems in strong (Theorem 3) and weak (Theorem 4) forms are established for the total and local particles numbers in a supercritical CBP. The results obtained generalize a number of previous ones proved, e.g., in papers [7] (Lemma 5.1) and [15]. So, limit distributions for particles numbers in the model of supercritical branching random walk on integer lattice $\mathbb{Z}^d$, $d \in \mathbb{N}$, with a finite number of particles generation centers are found in [15]. There not only existence of all moments of an offsprings number for any particle was assumed but also a certain rate of growth of such moments in dependence on its order. In our work, besides considering a more general model we also impose extremely weak restrictions on the moments of the offsprings size for each particle and, moreover, study the limit behavior of the particles numbers in the sense of almost surely convergence. These results are not only of self-contained interest but will be also applied in further investigations of the character of the particles population propagation in CBP. It is worthwhile to note that the mentioned achievements in the domain of limit theorems for supercritical CBP are due to the foundation laid in a recent work [4]. Firstly, there were introduced auxiliary Bellman-Harris branching processes which mainly allows us to reduce the study of CBP to classic results of modern theory of branching processes. Secondly, the moment analysis of the total and local particles numbers was implemented in [4] and the proofs of results of the present paper substantially bear on it.

Let us give a formal description of CBP. At the initial time $t = 0$ there is a single particle, its movement on a specified finite or countable set S is governed by continuous-time Markov chain $\eta = \{\eta(t), t \geq 0\}$ with infinitesimal matrix $Q = (q(x, y))_{x,y \in S}$. Hitting a finite catalysts set $W = \{w_1, \ldots, w_N\} \subset S$, e.g., at point w_k, the particle spends there random time distributed according to the exponential law with parameter $\beta_k > 0$. Then it either produces offsprings or leaves the element w_k with probabilities α_k and $1 - \alpha_k$ ($0 \leq \alpha_k < 1$), respectively. If the particle produces offsprings (at w_k), it dies instantly turning into a random number ξ_k of offsprings located at the same point w_k. If the particle leaves w_k, it jumps to point $y \neq w_k$ with probability $-q(w_k, y)q(w_k, w_k)^{-1}$ and continues the movement controlled by the Markov chain η. All the newborn particles behave themselves as independent copies of the parent particle. We assume that the Markov chain η is irreducible and the matrix Q is conservative, i.e. $\sum_{y \in S} q(x, y) = 0$ as well as $q(x, y) \geq 0$ for $x \neq y$ and $q(x, x) \in (-\infty, 0)$ for any $x \in S$. Denote by $f_k(s) := \mathsf{E}s^{\xi_k}$, $s \in [0, 1]$, the probability generating function of random variable ξ_k, $k = 1, \ldots, N$. We will employ the standard condition of the finite derivative $f_k'(1)$ existence,

i.e. of $\mathsf{E}\xi_k$ finiteness for each $k = 1, \ldots, N$. Let $\mu(t)$ be the total particles number existing in CBP at time $t \geq 0$ and the local particles numbers $\mu(t; y)$ are the quantities of particles located at separate points $y \in S$ at time t.

To formulate the main results of the paper let us introduce additional notation. We temporarily forget that there are catalysts at some elements of set S and consider only the movement of a particle on set S in accordance with Markov chain η with generator Q and starting point x. Let ${}_H\overline{\tau}_{x,y}$ be the time elapsed from the exit moment of this Markov chain out of starting state x till the first hitting point y by the particle whenever the particle trajectory does not pass the set $H \subset S$. Otherwise, we put ${}_H\overline{\tau}_{x,y} = \infty$. Extended random variable ${}_H\overline{\tau}_{x,y}$ is called *hitting time* of state y *under taboo* on set H after exit out of starting state x (see, e.g., [8], Part II, Section 11). Denote by ${}_H\overline{F}_{x,y}(t)$, $t \geq 0$, the improper cumulative distribution function of this extended random variable. Mainly we will be interested in the situation when $H = W_k$ where $W_k := W \setminus \{w_k\}$, $k = 1, \ldots, N$. Further

$$F^*(\lambda) := \int_{0-}^{\infty} e^{-\lambda t}\, dF(t), \quad \lambda \geq 0,$$

denotes the Laplace transform of cumulative distribution function $F(t)$, $t \geq 0$, with support located on non-negative semi-axis. For $j, k = 1, \ldots, N$ and $t \geq 0$ set $G_j(t) := 1 - e^{-\beta_j t}$,

$$G_{j,k}(t) := \beta_j \int_0^t {}_{W_k}\overline{F}_{w_j,w_k}(t-u) e^{-\beta_j u}\, du$$

and

$${}_{W_k}F_{x,w_k}(t) := -q(x,x) \int_0^t {}_{W_k}\overline{F}_{x,w_k}(t-u) e^{q(x,x)u}\, du.$$

At last, let

$${}_{W_k}\overline{F}_{x,w_k}(\infty) = {}_{W_k}F_{x,w_k}(\infty) := \lim_{t\to\infty} {}_{W_k}F_{x,w_k}(t).$$

In [4] there was introduced a matrix function $D(\lambda)$ which is irreducible matrix of size $N \times N$ for each $\lambda \geq 0$. Namely, $D(\lambda) = (d_{i,j}(\lambda))_{i,j=1}^N$ where

$$d_{i,j}(\lambda) = \delta_{i,j}\alpha_i f_i'(1) G_i^*(\lambda) + (1 - \alpha_i) G_i^*(\lambda) {}_{W_j}\overline{F}^*_{w_i,w_j}(\lambda)$$

and $\delta_{i,j}$ is the Kronecker delta. According to definition 1 in [4] CBP is called supercritical if the Perron root (i.e. positive eigenvalue being the spectral radius) $\rho(D(0))$ of the matrix $D(0)$ is greater than 1. Then in view of monotonicity of all elements of matrix function $D(\cdot)$ there exists the solution $\nu > 0$ of equation $\rho(D(\lambda)) = 1$. As Theorem 1 in [4] shows, just this positive number ν specifies the rate of exponential growth of the mean total and local particles numbers (in the literature devoted to population dynamics and branching processes one traditionally speaks of Malthusian parameter). More exactly, $\mathsf{E}_x\mu(t) \sim A(x)e^{\nu t}$ and $\mathsf{E}_x\mu(t; y) \sim a(x,y)e^{\nu t}$ as $t \to \infty$ where the index $x \in S$ stands for the starting point of CBP. The explicit formulae for functions $A(\cdot)$ and $a(\cdot,\cdot)$ are given in [4].

Theorem 1 is devoted to solution of the problem of finding the global extinction probability $q(x) := \mathsf{P}_x(\lim_{t\to\infty} \mu(t) = 0) = \lim_{t\to\infty} \mathsf{P}_x(\mu(t) = 0)$, $x \in S$, for the model of CBP.

Theorem 1 *For the global extinction probability $q(x)$ when $x \in S \setminus W$ there exists a representation*

$$q(x) = \sum_{k=1}^{N} {}_{W_k}F_{x,w_k}(\infty) q(w_k) \tag{1}$$

where the values $q(w_j)$, $j = 1, \ldots, N$, satisfy with the following system of equations

$$q(w_j) = \alpha_j f_j(q(w_j)) + (1-\alpha_j) \sum_{k=1}^{N} {}_{W_k}F_{w_j,w_k}(\infty) q(w_k). \tag{2}$$

In addition, vector $(q(w_1), \ldots, q(w_N))$ is component-wise the least root of equations system (2) in the cube $[0,1]^N$. Moreover, if the Markov chain η is recurrent then $q(x) = 1$, $x \in S$, or $q(x) < 1$, $x \in S$, whenever $\rho(D(0)) \leq 1$ or $\rho(D(0)) > 1$, respectively. If the Markov chain η is transient then $q(x) < 1$ for all $x \in S$.

One may oppose the global extinction probability $q(x)$ to the local extinction probability $Q(x,y) = \mathsf{P}_x \left(\limsup_{t\to\infty} \mu(t;y) = 0\right)$, for $x, y \in S$. The following theorem shows that in fact the function $Q(x,y)$ does not depend on variable y.

Theorem 2 *Equality $Q(x,y) = Q(x)$ holds for any $y \in S$ where the function $Q(x)$ when $x \in S \setminus W$ is of the form*

$$Q(x) = \sum_{k=1}^{N} {}_{W_k}F_{x,w_k}(\infty) Q(w_k) + 1 - \sum_{k=1}^{N} {}_{W_k}F_{x,w_k}(\infty) \tag{3}$$

and the values $Q(w_j)$, $j = 1, \ldots, N$, are the least solution to the equations system

$$\begin{aligned} Q(w_j) = \alpha_j f_j(Q(w_j)) \quad &+ \quad (1-\alpha_j) \sum_{k=1}^{N} {}_{W_k}F_{w_j,w_k}(\infty) Q(w_k) \\ &+ \quad (1-\alpha_j) \left(1 - \sum_{k=1}^{N} {}_{W_k}F_{w_j,w_k}(\infty)\right) \end{aligned} \tag{4}$$

in the cube $[0,1]^N$. Moreover, if $\rho(D(0)) \leq 1$ then $Q(x) = 1$ for all $x \in S$. Whenever $\rho(D(0)) > 1$ one has $Q(x) < 1$ for each $x \in S$.

Applying Theorems 4 and 5 in [12], Chapter 5, Section 1, to the auxiliary Bellman-Harris processes constructed in [4] we come to the statements of Theorems 1 and 2. Clearly $0 \leq q(x) \leq Q(x) \leq 1$, $x \in S$. In view of the explicit form of relations (1)-(4) we conclude that if the strict inequality $\sum_{k=1}^{N} {}_{W_k}F_{x,w_k}(\infty) < 1$ is satisfied at least for some $x \in S$, i.e. the Markov chain η is transient, then $q(x) < Q(x)$ for all $x \in S$. Otherwise, i.e. if the Markov chain η is recurrent, the relations for $q(\cdot)$ and $Q(\cdot)$ coincide and whence $q(x) = Q(x)$ for all x. In the terms of paper [2] the aforesaid means that, for transient Markov chain η and $\rho(D(0)) \leq 1$, we deal with the pure global survival phase of CBP and in the case of recurrent Markov chain η and $\rho(D(0)) > 1$ the strong local survival of CBP is observed.

Now we pass to considering the problem of the population growth rate in the case of global and local survival, i.e. whenever $\rho(D(0)) > 1$. Let $\mathbf{u} = (u_1, \ldots, u_N)$ be the right eigenvector of matrix $D(\nu)$ corresponding to the Perron root $\rho(D(\nu))$ equal to 1 where $u_k > 0$, $k = 1, \ldots, N$, and $\sum_{k=1}^{N} u_k = 1$. It should be noted that, by virtue of the Perron-Frobenius theorem (see, e.g., [12], Chapter IV, Section 5), such eigenvector can always be found since the matrix $D(\nu)$ is irreducible according to Lemma 1 in [4]. Recall the definition of matrix function $D(x;\lambda)$, $x \notin W$, $\lambda \geq 0$, introduced in [4] while proving Theorem 1, case 2. For $x \notin W$, set

$w_{N+1} = x$, $W(x) := W \cup \{x\}$ and $W_i(x) := W(x) \setminus \{w_i\}$, $i = 1, \ldots, N+1$. Then the matrix $D(x;\lambda) = (d_{i,j}(x;\lambda))_{i,j=1}^{N+1}$ has elements

$$d_{i,j}(x;\lambda) := \delta_{i,j}\, \alpha_i f_i'(1) G_i^*(\lambda) + (1-\alpha_i) G_i^*(\lambda)_{W_j(x)} \overline{F}^*_{w_i,w_j}(\lambda), \quad \lambda \geq 0.$$

Here $\alpha_{N+1} = 0$, $f_{N+1}'(1) = 0$ and $G_{N+1}(t) := 1 - e^{q(x,x)t}$, $t \geq 0$. By Lemma 3 in [4] the matrix $D(x;\lambda)$ is irreducible and $\rho(D(x;\nu)) = 1$. Define $\mathbf{u}(x) = (u_1(x), \ldots, u_{N+1}(x))$ to be the right eigenvector of the matrix $D(x;\nu)$, corresponding to the Perron root $\rho(D(x;\nu))$ equal to 1, such that $u_k(x) > 0$, $k = 1, \ldots, N+1$, and $\sum_{k=1}^N u_k(x) = 1$. Take notice that bearing on the proof of Lemma 3 in [4] it is not difficult to verify equalities $u_i(x) = u_i$ for all $i = 1, \ldots, N$ and $x \in S \setminus W$. Set $c(x) := u_k^{-1}$ whenever $x = w_k$ for some $k = 1, \ldots, N$ and $c(x) := u_{N+1}^{-1}(x)$ whenever $x \in S \setminus W$. Let also symbols $\mathbf{0}$ and $\mathbf{1}$ denote vectors with zeros and units, respectively, as all components, the dimension of vectors being contextually clear.

In the given paper, we handle three forms of random variables convergence, viz., almost surely, in probability and in distribution which are denoted by $\xrightarrow{a.s.}$, $\xrightarrow{\mathsf{P}}$ and $\xrightarrow{d}$, respectively. Theorem 3 describes strong convergence of vectors of the normalized total and local particles numbers in CBP as time t grows to infinity. The results on asymptotic behavior of the normalizing means obtained in Theorem 1 in [4] are implicitly employed in the proofs of Theorems 3 and 4.

Theorem 3 *Let a supercritical CBP start at point $x \in S$. Assume that the elements of matrix Q are uniformly bounded, i.e. for all $z_1, z_2 \in S$ and some constant $C > 0$ one has $|q(z_1, z_2)| < C$. Let also $\mathsf{E}\xi_k^2 < \infty$ for each $k = 1, \ldots, N$. Then there exists non-degenerate random variable ζ such that for any $n \in \mathbb{N}$ and $y_1, \ldots, y_n \in S$ the following relation is valid*

$$\left(\frac{\mu(t)}{\mathsf{E}_x \mu(t)}, \frac{\mu(t;y_1)}{\mathsf{E}_x \mu(t;y_1)}, \ldots, \frac{\mu(t;y_n)}{\mathsf{E}_x \mu(t;y_n)} \right) \xrightarrow{a.s.} c(x)\zeta\mathbf{1}, \quad t \to \infty. \tag{5}$$

If the Markov chain η is recurrent, the assertion of Theorem 3 ensues from the construction of auxiliary Bellman-Harris branching process in [4] and applying to it Theorem 4.1 in [11] with due regard of the fact that under the conditions of Theorem 3 the distributions of life-lengths of particles of different types in the auxiliary Bellman-Harris process are absolutely continuous and have bounded densities. The latter observation is true on account of the argument in the final part of paper [5].

If the Markov chain η is transient then one has to apply the proof scheme of Theorem 4.1 in [11] to the auxiliary Bellman-Harris process with final type of particles which was introduced in [4] while establishing the case 1 (step 6) of Theorem 1.

Now we relax the restrictions on the studied process. For $z \in \mathbb{R}$, set $\log^+ z := \log(max\{z, 1\})$.

Theorem 4 *Let a supercritical CBP start at point $x \in S$. Then for each $n \in \mathbb{N}$ and any $y_1, \ldots, y_n \in S$ the following alternative is true.*

1. *If $\mathsf{E}\xi_k \log^+ \xi_k = \infty$ for some $k \in \{1, \ldots, N\}$ then*

$$\left(\frac{\mu(t)}{\mathsf{E}_x \mu(t)}, \frac{\mu(t;y_1)}{\mathsf{E}_x \mu(t;y_1)}, \ldots, \frac{\mu(t;y_n)}{\mathsf{E}_x \mu(t;y_n)} \right) \xrightarrow{\mathsf{P}} \mathbf{0}, \quad t \to \infty. \tag{6}$$

2. *If $\mathsf{E}\xi_k \log^+ \xi_k < \infty$ for all $k = 1, \ldots, N$ then*

$$\left(\frac{\mu(t)}{\mathsf{E}_x \mu(t)}, \frac{\mu(t;y_1)}{\mathsf{E}_x \mu(t;y_1)}, \ldots, \frac{\mu(t;y_n)}{\mathsf{E}_x \mu(t;y_n)} \right) \xrightarrow{d} c(x)\zeta\mathbf{1}, \quad t \to \infty. \tag{7}$$

Here ζ is a non-degenerate random variable with the following properties.

(i) $\mathsf{E}_x\zeta = c(x)^{-1}$.

(ii) $\mathsf{P}_x(\zeta = 0) = \mathsf{P}_x\left(\limsup_{t\to\infty} \mu(t; y) = 0\right)$ *for any* $y \in S$.

(iii) *The Laplace transform* $\varphi(\lambda; x) := \mathsf{E}_x e^{-\lambda\zeta}$, $\lambda \geq 0$, $x \in S$, *of random variable* ζ *for* $x \in S\backslash W$ *is of the form*

$$\varphi(\lambda; x) = \sum_{k=1}^{N} \int_0^\infty \varphi(\lambda e^{-\nu u}; w_k)\, d_{W_k}F_{x,w_k}(u) + 1 - \sum_{k=1}^{N} {}_{W_k}F_{x,w_k}(\infty)$$

where functions $\varphi(\cdot; w_j)$, $j = 1, \ldots, N$, *satisfy the system of integral equations*

$$\begin{aligned}
\varphi(\lambda; w_j) &= \alpha_j \int_0^\infty f_j(\varphi(\lambda e^{-\nu u}; w_j))\, dG_j(u) \\
&+ (1-\alpha_j) \sum_{k=1}^{N} \int_0^\infty \varphi(\lambda e^{-\nu u}; w_k)\, dG_{j,k}(u) \\
&+ (1-\alpha_j)\left(1 - \sum_{k=1}^{N} {}_{W_k}\overline{F}_{w_j,w_k}(\infty)\right).
\end{aligned}$$

(iv) *The conditional distribution of* ζ *under condition of CBP start at point* $x \in S$ *is absolutely continuous on the positive semi-axis and has continuous density function.*

Note that in relation (7) of Theorem 4 exactly the convergence of vectors is essential whereas in formulae (5) and (6) the vectors convergence is tantamount to convergence of their components.

If the Markov chain η is recurrent then Theorem 4 is proven with the help of Theorem 1.1 in [10], applied to the constructed in [4] auxiliary Bellman-Harris process. However, if the Markov chain η is transient, the statement of Theorem 4 is established by applying the proof scheme of Theorem 1.1 in [10] to the Bellman-Harris process with final type of particles introduced while proving the case 1 (step 6) of Theorem 1 in [4].

Thus, under weak conditions new asymptotic properties of CBP are investigated.

References

[1] Albeverio S., Bogachev L.V., Yarovaya E.B. Asymptotics of branching symmetric random walk on the lattice with a single source. *C. R. Math. Acad. Sci. Paris*, 326(9), 1998, 975-980.

[2] Bertacchi D., Zucca F. Strong local survival of branching random walks is not monotone. *Adv. in Appl. Probab.*, 46(2), 2014, 400-421.

[3] Bocharov S., Harris S.C. Branching Brownian motion with catalytic branching at the origin. *Acta Appl. Math.*, 134(1), 2014, 201-228.

[4] Bulinskaya E.Vl. Complete classification of catalytic branching processes. *Teor. Veroyatnost. i Primenen.*, 59(4), 2014, 639-666.

[5] Bulinskaya E.Vl. Finiteness of hitting times under taboo. *Statist. Probab. Lett.*, 85(1), 2014, 15-19.

[6] Bulinskaya E.Vl. Local particles numbers in critical branching random walk. *J. Theoret. Probab.*, 27(3), 2014, 878-898.

[7] Carmona Ph., Hu Y. The spread of a catalytic branching random walk. *Ann. Inst. Henri Poincaré Probab. Stat.*, 50(2), 2014, 327-351.

[8] Chung K.L. *Markov chains with stationary transition probabilities.* Springer, Berlin, 1960.

[9] Doering L., Roberts M. Catalytic branching processes via spine techniques and renewal theory. In: Donati-Martin C., et al. (Eds.), *Séminaire de Probabilités* XLV, *Lecture Notes in Math.*, 2078, 2013, 305-322.

[10] Kaplan N. The supercritical multitype age-dependent branching process. *J. Math. Anal. Appl.*, 50(1), 1975, 164-182.

[11] Mode Ch.J. A multidimensional age-dependent branching process with applications to natural selection. II. *Math. Biosci.*, 3, 1968, 231-247.

[12] Sewastianow B.A. *Verzweigungsprozesse.* Akademie, Berlin, 1974 (in German).

[13] Topchii V.A., Vatutin V.A. Catalytic branching random walks in $\mathbb{Z}^d$ with branching at the origin only. *Siberian Adv. Math.*, 23(2), 2013, 123-153.

[14] Vatutin V.A. *Branching processes and their applications.* Lektsionnye Kursy NOC (Lectures of the Educational Center of Steklov Mathematical Institute). Moscow, MIRAS, 2009, No. 8 (in Russian).

[15] Yarovaya E.B. Spectral properties of evolutionary operators in branching random walk models. *Math. Notes*, 92(1), 2012, 115-131.